MATH 420 HW #4 SOLUTIONS

Instructions: A printed copy of your homework should be handed in at **the start of class** the day it is due. Supplementary electronic files (e.g. R scripts or wxMaxima files;) should be emailed to the instructor prior to class with file name format LASTNAME-HWX.EXT

Your assignment is to provide a nicely written-up derivation of the system of ODEs below starting from a discrete-time stochastic model.

Background: Suppose there are N_0 atoms of radioactive Uranium. Over time interval Δt each can decay with probability (w.p.) $\lambda\Delta t$.

Let $U(t)$ be the number of Uranium atoms. The number lost during time interval $[t, t + \Delta t]$ is approximately a binomial *random variable* with parameters $n = U(t)$ and $p = \lambda\Delta t$. Thus, the *expected number* lost is $n\,p = \lambda\,U(t)\,\Delta t$.

Assuming U_0 is large, then the Law of Large Numbers (LLN) allows us to claim $U(t + \Delta t) - U(t) \approx -\lambda\,U(t)\,\Delta t$. Taking $\Delta t \to 0$ we can derive the **mean field** model:

$$\frac{dU(t)}{dt} = -\lambda U(t), \qquad U(0) = U_0$$

Now instead suppose there are U_0 atoms of radioactive Uranium, and these can decompose into Thorium and then again decompose into Protactinium. Specifically, over time interval Δt, Uranium atoms can decay w.p. $\lambda_\alpha\Delta t$ to Thorium and an α particle ${}^4_2\mathrm{He}$. Thorium can then decay via loss of a β particle (positron) to Protactinium w.p. $\lambda_\beta\Delta t$.

Let $T(t)$ be the number of thorium atoms, and $P(t)$ the number of protactinium atoms.

Assignment: Derive the following UTP model

$$\begin{aligned}
\frac{dU(t)}{dt} &= -\,\lambda_\alpha U(t) \\
\frac{dT(t)}{dt} &= \lambda_\alpha U(t) - \lambda_\beta T(t) \\
\frac{dP(t)}{dt} &= \lambda_\beta T(t)
\end{aligned}$$

1. Write a (stochastic) discrete time map (step size Δt) that models the numbers of atoms transitioning states in each time step using Binomial distributions.

   Ans: First, let $\widetilde{U}(t)$, $\widetilde{T}(t)$ and $\widetilde{P}(t)$ be the stochastic variables representing the amounts of Uranium, Thorium and Protactinium at time t. Let $X_1(t)$ be the number of atoms at time t that transition from Uranium to Thorium in $(t, t + \Delta t]$ and let $X_2(t)$ be the number of atoms at time t that transition from Thorium to Protactinium in $(t, t + \Delta t]$.

Then $X_1(t) \sim$Binomial$(\widetilde{U}(t), \lambda_\alpha \Delta t)$ and $X_2(t) \sim$Binomial$(\widetilde{P}(t), \lambda_\beta \Delta t)$, and

$$\begin{aligned}
\widetilde{U}(t+\Delta t) =& \widetilde{U}(t) - X_1(t) \\
\widetilde{T}(t+\Delta t) =& \widetilde{P}(t) + X_1(t) - X_2(t) \\
\widetilde{P}(t+\Delta t) =& \widetilde{T}(t) + X_2(t)
\end{aligned}$$

2. Use the LLN to find the corresponding mean-field map.

Ans: If we let $U(t)$, $T(t)$, and $P(t)$ be the expected values of $\widetilde{U}(t)|\widetilde{U}(t-\Delta t)$, $\widetilde{T}(t)|\widetilde{T}(t-\Delta t)$ and $\widetilde{P}(t)|\widetilde{P}(t-\Delta t)$, respectively, then taking the expected value of the above equations and approximating using the Law of Large Numbers yields

$$\begin{aligned}
U(t+\Delta t) =& U(t) - \lambda_\alpha \Delta t U(t) \\
T(t+\Delta t) =& P(t) + \lambda_\alpha \Delta t U(t) - \lambda_\beta \Delta t T(t) \\
P(t+\Delta t) =& T(t) + \lambda_\beta \Delta t T(t)
\end{aligned}$$

3. Take the limit as $\Delta t \to 0$ to find the continuous time (ODE) approximation of this mean-field discrete map, i.e., the above system of ODEs.

Ans: Rearranging the above equations yields

$$\begin{aligned}
\frac{U(t+\Delta t) - U(t)}{\Delta t} =& -\lambda_\alpha U(t) \\
\frac{T(t+\Delta t) - P(t)}{\Delta t} =& \lambda_\alpha U(t) - \lambda_\beta T(t) \\
\frac{P(t+\Delta t) - T(t)}{\Delta t} =& \lambda_\beta T(t)
\end{aligned}$$

Taking the limit as $\Delta t \to 0$ yields the desired UTP model.