Maps: Discrete-Time Dynamical Systems
Week 12 – Monday

Mathematical Modeling (Math 420/620)

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Equilibria & Local Stability

Comparison of Equilbrium Stability Analysis for continuous- (ODE) and discrete-time (maps) dynamical systems.

ODEs

Assume $t \in [0, \infty)$, $\mathbf{x}(t) \in \mathbb{R}^n$, f differentiable.

$$\dot{\mathbf{x}} = f(x)$$

Solutions: Continuous $\mathbf{x}(t)$

Maps

Assume $t \in \{0, 1, 2 \ldots\}$, $\mathbf{x}_k \in \mathbb{R}^n$, f differentiable.

$$x_{t+1} = f(x_t)$$

Solutions: $\mathbf{x}_0, \mathbf{x}_1, \ldots$

Equilibria

ODEs

Equilibria $\mathbf{x}_*$ satisfy $\dot{\mathbf{x}} = 0$, and thus satisfy

$$f(\mathbf{x}_*) = 0.$$

Example:

$$\dot{x} = r\,x$$

implies

$$x_* = 0$$

Maps

Equilibria (aka **fixed points**) satisfy $\mathbf{x}_{k+1} = \mathbf{x}_k$, and thus

$$f(\mathbf{x}_*) = \mathbf{x}_*.$$

Example:

$$x_{k+1} = r\,x_k$$

implies

$$x_* = 0$$

Note: In the ODE example, r is an *exponential growth* ($r > 0$) or *decay* ($r < 0$) *rate*. In the discrete map example, it is a *multiplier*.

Local Asymptotic Stability (LAS)

ODEs: $\dot{\mathbf{x}} = f(\mathbf{x})$

Equilibrium $\mathbf{x}_*$ is LAS if the Jacobian of f evaluated at $\mathbf{x}_*$ has all **eigenvalues** with **negative real part**:

$$\mathrm{Re}(\lambda_i) < 0$$

Example:

$$\dot{x} = r\,x$$

$$x_* = 0, \quad \lambda = r$$

Thus, x_* is **stable** if $r < 0$.

Maps: $\mathbf{x_{n+1}} = f(\mathbf{x}_n)$

Equilibrium $\mathbf{x}_*$ is LAS if the Jacobian of f evaluated at $\mathbf{x}_*$ has all **eigenvalues** with **magnitude** $<$ **1**:

$$|\lambda_i| < 1$$

Example:

$$x_{k+1} = r\,x_k$$

$$x_* = 0, \quad \lambda = r$$

Thus, x_* **stable** if $r \in (-1, 1)$.