Stochastic Models
Week 11 – Wednesday

Mathematical Modeling (Math 420/620)

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Announcements

http://www.pauljhurtado.com/teaching/FA15/

No Class Next Wednesday (11/11)

- Veterans Day

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Exam Options

- in-class: Nov 18? Nov 23?

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- in-class: Nov 18? Nov 23?
- take-home Nov 16-23? Nov 12-20? Other?

Markov Chains

Markov chain: Consider a sequence of r.v.s $X_k \in \{1, ..., N\}$ where $X_0 = x_0$ is some fixed constant. Then for $k = 1, 2,$... the conditional probability

$$P(X_{k+1} = j | X_k = i) = p_{ij}$$

We call the matrix $\mathbf{P} = (p_{ij})$ the **transition matrix** of the Markov Chain.

Markov Chains

Let the elements of vector π be the probabilities of being in state i after k “jumps” (iterations).

$$\pi_k(i) = P(X_k = i)$$

Then

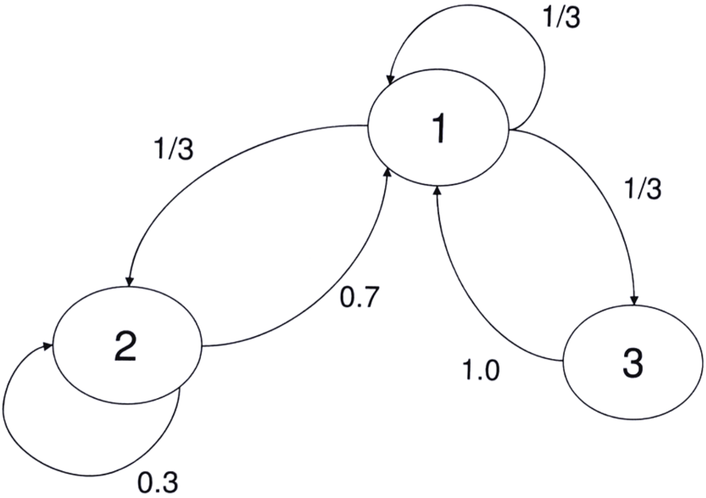
$$\pi_{k+1} = \pi_k \, \mathbf{P}$$

and when there exist a solution to

$$\pi = \pi \, \mathbf{P}$$

we say the stochastic process has a steady state π.

Example (Fig. 8.2)



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Transition Matrix?

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Thus,

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0.7 & 0.3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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Exercise #1: Find π_2 if $\pi_1 = [1/3, 1/3, 1/3]$.

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Exercise #1: Find π_2 if $\pi_1 = [1/3, 1/3, 1/3]$.

Exercise #2: Does π_k converge as $k \to \infty$?

Theorem (Perron-Frobenius)

If a non-negative, square matrix $\mathbf{P}$ *raised some some power* m *yields a matrix* $\mathbf{P}^m$ *which has strictly positive entries (i.e.,* $\mathbf{P}$ *is power-positive) then* $\mathbf{P}$ *has a unique dominant eigenvalue which is real and positive, and the corresponding eigenvector has all positive entries.*

Furthermore, an nxn non-negative matrix $\mathbf{P}$ is power-positive *if and only if* $\mathbf{P}^{n^2-2n+2}$ has strictly positive entries.