Dynamical Systems: Introduction

Mathematical Modeling (Math 420/620)

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November 8, 2017

Overview

Building Dynamic Models, ODEs

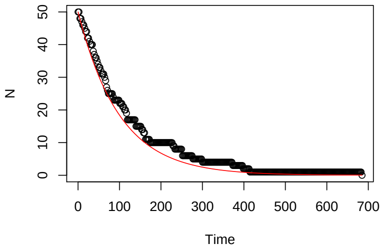
- Mean Field Equations & "Bathtub" Models

Analysis of Dynamic Models (Topic Overview)

- State Space & Vector Fields
- Asymptotic Behavior: What happens as $t \to \infty$? Parameter dependence?
- Equilibrium Stability Analysis
- Other dynamics? Bifurcation Theory
- Other Attractors: Limit Cycles, etc.
- Sensitivity Analysis & Simulation

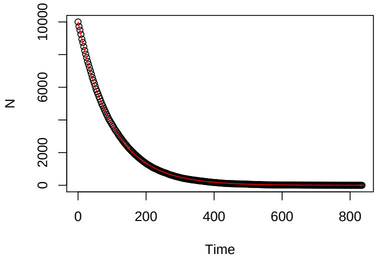
Example: Exponential Decay

```
## Ex: tracking atoms experiencing radioactive decay
Ts=sort(rexp(50,1/100))
Time=seq(0,max(Ts),length=300)
N=Time*0; # counts of atoms at time t go here.
N[1]=50;
for(i in 2:300) { N[i]=sum(Ts > Time[i]) } # number not yet decayed
plot(Time,N);  curve(50*(exp(-x/100)),0,max(Ts),add=TRUE,col="red")
```



Example: Exponential Decay

```
## Ex: tracking atoms experiencing radioactive decay
Ts=sort(rexp(1e4,1/100))
Time=seq(0,max(Ts),length=300)
N=Time*0; # counts of atoms at time t go here.
N[1]=1e4;
for(i in 2:300) { N[i]=sum(Ts > Time[i]) } # number not yet decayed
plot(Time,N);  curve(1e4*(exp(-x/100)),0,max(Ts),add=TRUE,col="red")
```

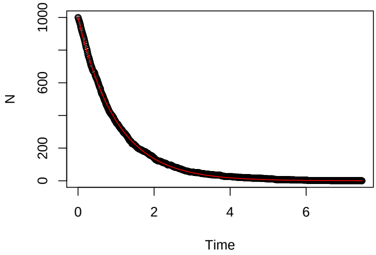


Example: Exponential Decay as Stochastic Map

Simulate as a discrete map with a Binomial # of atoms decaying each time step, i.e.,

$$N(t+dt) = N(t) - rbinom(1, n = N(t), prob = r * dt)$$

```
N0=1e3; dt=1/100; r=1; N=c(N0); i=1;
while(N[i] > 0) { N[i+1]=N[i]-rbinom(1,N[i],r*dt); i=i+1; } # number no
Time=dt*(1:length(N))-dt; plot(Time,N,xlab="Time");
curve(N0*(exp(-r*x)),0,dt*length(N),add=TRUE,col="red")
```



Implicit Assumptions?

Which (implicit) assumptions were made? Which could be relaxed?

Spatially structured interactions?

Small N vs $N \to \infty$?

Time-dependent or N-dependent rate?

Others?

Good rule of thumb with ODE models:
Implicit assumptions typically ignore spatial interactions, stochastic variation and/or small numbers of individuals, and/or the discrete nature of individuals.

Mean Field Equations: Applying LLN, CLT

Example: Suppose there are N_0 atoms of radioactive ${}^{238}_{92}\text{U}$. Over time interval Δt each can decay w.p. $\lambda \Delta t$.

Let $N(t)$ be the number of uranium atoms. The number lost during time interval $[t, t + \Delta t]$ is approximately a binomial *random variable* with parameters $n = N(t)$ and $p = \lambda \Delta t$. Thus, the *expected number* lost is $n\,p = \lambda\, N(t)\, \Delta t$.

Assuming N_0 is large, then the Law of Large Numbers (LLN) allows us to claim $N(t + \Delta t) - N(t) \approx -\lambda\, N(t)\, \Delta t$. Taking $\Delta t \to 0$ we can derive the **mean field** model:

$$\frac{dN(t)}{dt} = -\lambda N(t), \qquad N(0) = N_0$$

Mean Field Equations

Example: Suppose there are U_0 atoms of radioactive $^{238}_{92}\text{U}$. Over time interval Δt each can decay w.p. $\lambda_\alpha \Delta t$ to $^{234}_{90}\text{Th}$ and α particle $^{4}_{2}\text{He}$. Thorium-234 can then decay via loss of a β particle (positron) to protactinium-234 w.p. $\lambda_\beta \Delta t$.

Let $T(t)$ be the number of thorium atoms, and $P(t)$ the number of protactinium atoms. We can now use the model

$$\begin{aligned}
\frac{dU(t)}{dt} &= -\lambda_\alpha U(t) \\
\frac{dT(t)}{dt} &= \lambda_\alpha U(t) - \lambda_\beta T(t) \\
\frac{dP(t)}{dt} &= \lambda_\beta T(t)
\end{aligned}$$

Exercise

Derive the following UTP model

$$\begin{aligned}\frac{dU(t)}{dt} &= -\lambda_\alpha U(t)\\ \frac{dT(t)}{dt} &= \lambda_\alpha U(t) - \lambda_\beta T(t)\\ \frac{dP(t)}{dt} &= \lambda_\beta T(t)\end{aligned}$$

1. Write a discrete time map (step size Δt) that models the numbers of atoms transitioning states in each time step using Binomial distributions.
2. Use the LLN to find the corresponding mean-field map.
3. Take the limit as $\Delta t \to 0$ to find the continuous time (ODE) approximation of this mean-field discrete map.

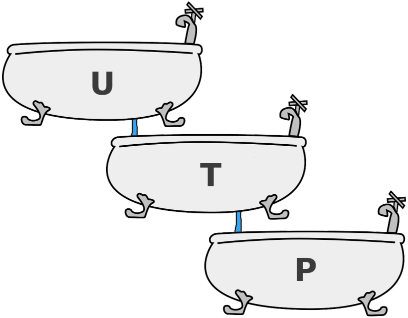
ODEs: "Bathtub" Models

Model the "flow" of mass from one compartment to another:

$$\frac{dU(t)}{dt} = -\lambda_\alpha U(t)$$

$$\frac{dT(t)}{dt} = \lambda_\alpha U(t) - \lambda_\beta T(t)$$

$$\frac{dP(t)}{dt} = \lambda_\beta T(t)$$



Intuition for ODE model terms

- **Recall the 5-step process!**

  Question? Assumptions? Simplify, etc...

- ODE models often *average* over heterogeneity, space, etc.
- **Linear terms** correspond to **exponential decay** rates.
- More complex transition rates? Derive[1] terms accordingly.

[1] Remember: Lie, Cheat, Steal! (see Ch. 9 in Ellner & Guckenheimer)

Dynamic Model (ODE) Basics

Suppose $\mathbf{x} \in \mathbb{R}^n$, functions $f = [f_1, f_2, \ldots, f_n]$ are *smooth*[2], and

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x_0}.$$

State Variables: $\mathbf{x} = [x_1, x_2, \ldots, x_n]$
Initial Conditions: $\mathbf{x_0}$
State Space: $S \subseteq \mathbb{R}^n$ ($n = \#$ of state var.)
Vector Field: f
Parameter Space: Ex: $\mathbb{R}^{n^2}$ for a full linear system.
Trajectory/Orbit: Solutions $\mathbf{x}(t)$ to the above IVP.

[2] Continuous partial derivatives near $\mathbf{x_0}$ guarantee existence, uniqueness of solutions.

Examples

What are the state variables? State space? Parameter space?

1.

$$\frac{dx}{dt} = r\,x\,(1 - x/K)$$

2.

$$\frac{dN}{dt} = r\,N\left(1 - (N/K)^{\theta}\right)$$

3.

$$\frac{du}{d\tau} = u\,(1 - u^{\theta})$$

4.

$$\begin{aligned}\dot{H} &= r_H\,H - a_H\,H^2 - b_H\,S\,H\\ \dot{S} &= r_S\,S - a_S\,S^2 - b_S\,H\,S\end{aligned}$$

Equilibria

Trajectories are often categorized by **qualitative properties** (e.g. steady-state vs. cycling vs. chaos) of their **asymptotic behavior** (i.e., what do solutions look like as $t \to \infty$?).

Equilibrium solutions are the natural place to begin studying those asymptotic properties.

Definition

An **equilibrium** of

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

is any *constant* solution $\mathbf{x}(t) = \mathbf{x}_*$ which therefore satisfies

$$f(\mathbf{x}_*) = 0.$$

Equilibria

Find all equilibrium solutions to each of the following ODEs:

1. $$\frac{dN}{dt} = r\,N$$
2. $$\frac{dx}{dt} = K - x$$
3. $$\frac{dx}{dt} = x\,(K - x)$$
4. $$\frac{dx}{dt} = r\,x\,(1 - \frac{x}{K})$$
5. $$\frac{dx}{dt} = x\,(1 - x)(a - x)$$
6. $$\frac{dx}{dt} = \sin(x)$$

Stability Concepts

1. We say $\mathbf{x}_*$ is **locally asymptotically stable (LAS)** (or sometimes just *locally stable* or *attracting*) if all nearby trajectories converge to $\mathbf{x}_*$ (i.e., $\mathbf{x}(t) \to \mathbf{x}_*$ as $t \to \infty$).

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4. We call $\mathbf{x}_*$ **neutrally stable** if it is Lyapunov Stable but not attracting.

Phase Space & 1-D Vector Fields

Phase Space: Horizontal axis x, vertical axis $\frac{dx}{dt}$.

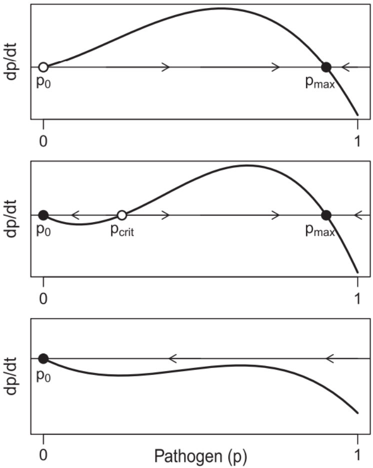
Bacterial infection growth model from Hurtado, 2012.

$$\frac{dp}{dt} = r\,p\,(1-p) - \frac{k\,p}{\mu + p}.$$

In the figures,

$$p_{crit} = \frac{\left((1-\mu) + \sqrt{(1+\mu)^2 - \frac{4}{r}k}\right)}{2}$$

$$p_{max} = \frac{\left((1-\mu) + \sqrt{(1+\mu)^2 - \frac{4}{r}k}\right)}{2}.$$



Equilibrium Stability

Theorem

(1D) An **equilibrium** x_* *of* $\dot{x} = f(x)$ *is* **locally asymptotically stable** *if*

$$f'(x_*) < 0$$

and is unstable if

$$f'(x_*) > 0.$$

Equilibrium Stability

Theorem

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Sketch of Proof.

If $u = x - x_*$, and f is smooth near x_* then $u = 0$ is an equilbrium of $\dot{u} \approx f'(x_*)\,u$ which has (approximately) exponential solutions that grow away from (or decay towards) 0 depending on the sign of $f'(x_*)$. □

Phase Space & 1-D Vector Fields

Sketch the *phase portrait* for each of the following, and use it to determine the stability of each equilibrium point:

1. $\frac{dx}{dt} = K - x$
2. $\frac{dx}{dt} = x\,(K - x)$
3. $\frac{dx}{dt} = r\,x\,(1 - \frac{x}{K})$
4. $\frac{dx}{dt} = x\,(1 - x)(a - x)$
5. $\frac{dx}{dt} = \sin(x)$

Equilibrium Stability

Theorem

An **equilibrium** $\mathbf{x}_*$ *of* $\dot{\mathbf{x}} = f(\mathbf{x})$ *is* **locally asymptotically stable (LAS)** *if the Jacobian matrix* $\mathbf{J}$ *(where* $J_{ij} = \frac{\delta f_i}{\delta x_j}$*) evaluated at* $\mathbf{x}_*$ *has eigenvalues with negative real parts. That is,* x_* *is LAS if* $Re(\lambda_i) < 0$ *for each of the* n *eigenvalues of matrix* $\mathbf{J}(\mathbf{x}_*)$*.*

Equilibrium Stability

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Sketch of Proof:

Consider the linear approximation of the vector field around $\mathbf{x}_*$. Then for a small neighborhood of $\mathbf{x}_*$,

$$\dot{\mathbf{x}} = f(\mathbf{x}) \approx \mathbf{J}(\mathbf{x}_*)\,\mathbf{x}.$$

Let $\mathbf{u} = \mathbf{x} - \mathbf{x}_*$ and $\mathbf{A} = \mathbf{J}(\mathbf{x}_*)$, then

$$\dot{\mathbf{u}} \approx \mathbf{A}\,\mathbf{u}$$

If $\mathbf{A}$ is full rank then ...

Sketch of Proof (cont'd):
... let $\mathbf{Q}$ be the matrix whose columns are the eigenvectors of $\mathbf{A}$, and let $\mathbf{D} = (\lambda_1, ..., \lambda_n)$. Then doing a standard change-of-coordinates

$$\begin{aligned} \dot{\mathbf{u}} &= \mathbf{Q}\,\mathbf{D}\,\mathbf{Q}^{-1}\,\mathbf{u} \\ \mathbf{Q}^{-1}\,\dot{\mathbf{u}} &= \mathbf{D}\,\mathbf{Q}^{-1}\,\mathbf{u} \\ \dot{\mathbf{y}} &= \mathbf{D}\,\mathbf{y} \end{aligned}$$

which implies $\dot{y}_i = \lambda_i\, y_i$ and thus

$$y_i(t) = y_i(0)\, \exp(\lambda_i\, t).$$

Therefore, **trajectories that begin sufficiently close to equilibrium $\mathbf{x}_*$ will approximately grow or decay at rate** $Re(\lambda_i)$ **along the corresponding eigenvectors of $\mathbf{J}(\mathbf{x}_*)$**.

Equilibrium Stability

Find all equilibrium solutions to each of the following ODEs:

$$1. \quad \frac{dx}{dt} = K - x$$

$$2. \quad \frac{dx}{dt} = x(1 - x)(a - x)$$

Two-species Competition (MMM Ex. 4.1)

$$\dot{H} = r_H\, H - a_H\, H^2 - b_H\, S\, H$$

$$\dot{S} = r_S\, S - a_S\, S^2 - b_S\, H\, S$$

Predator-Prey

$$\dot{x} = r\, x\, (1 - x) - \frac{a\, x\, y}{k + x}$$

$$\dot{y} = \frac{a\, x\, y}{k + x} - y$$

Two-species Competition

Goal: When can the two tree species coexist?

$$\dot{H} = r_H\, H - a_H\, H^2 - b_H\, S\, H$$
$$\dot{S} = r_S\, S - a_S\, S^2 - b_S\, H\, S$$

State Variables: (State Space is non-negative orthant in $\mathbb{R}^2$)

$H(t)$, $S(t)$ - Hardwood & Softwood population size (tons/acre)

Rates: (Units are tons/acre/year)

$g_H(t) = r_H\, H - a_H\, H^2$ Hardwood growth rate
$g_S(t) = r_S\, S - a_S\, S^2$ Softwood growth rate
$c_H(t) = b_H\, S\, H$ - Competitive impact on Hardwoods
$c_S(t) = b_S\, S\, H$ - Competitive impact on Softwoods

Parameters: intrinsic growth rate r_i, *intraspecific* competition coefficients a_i, and *interspecific* competition coefficient b_i.

Overview: Dynamic Models (ODEs)

Let

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x_0}$$

where $\mathbf{x}(t) \in \mathbb{R}^n \; \forall t \in \mathbb{R}$, and f is smooth.

Common Question in Applications:
What are the asymptotic dynamics of this model?

Approach:
(1) Equilibrium Stability Analysis and
(2) Bifurcation Analysis[3]

[3] We'll only briefly see bifurcation theory in this course. For more on the subject, I highly recommend Dynamical Systems & Chaos by Steve Strogatz.