Nondimensionalization Example

Here we nondimensionalize the system of equations

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2.$$

Note that there are two state variables, one time variable, and six parameters. Scaling both state variables and time should reduce the number of parameters by three.
Writing out the units of each symbol (here we use i, j to denote 1,2 or 2,1), we have

r_i	per unit time
N_i	# individuals
K_i	# individuals
b_i	per unit time, per # of individuals

The easiest way to non-dimensionalize N_i is to scale them each by K_i, therefore divide both sides of our equations by K_i and gather all N_i/K_i terms, substituting $n_i = N_i/K_i$ to get

$$\dot{n_i} = r_i n_i(1 - n_i) - B_i n_i n_j$$

We multiplied the last term by K_j/K_j to scale both $N_1 \cdot N_2$, thus we defined $B_i = b_i k_j$.

We now have the intermediate quantities

r_i	per unit time
n_i	unitless (interpret as proportion of individuals relative to K_i)
B_i	per unit time

Next, we scale time by one of the remaining rates, and choose one of the r_i rates because these will lead to easier interpretation than using say B_1 in an application context. Thus, dividing both equations by r_1 yields

$$\frac{dn_1}{r_1\,dt} = n_1(1 - n_1) - \frac{B_{12}}{r_1} n_1 n_2$$
$$\frac{dn_2}{r_1\,dt} = \frac{r_2}{r_1} n_2(1 - n_2) - \frac{B_{21}}{r_1} n_1 n_2$$

Set $\alpha = \frac{B_{12}}{r_1}$, $\beta = \frac{B_{21}}{r_1}$, $\gamma = \frac{r_1}{r_2}$ and $\tau = tr_1$. Then by the linearity of the derivative $r_1\,dt = d\tau$ and we have two dimensionless equations with three parameters (reduced from six!):

$$\frac{dn_1}{d\tau} = n_1(1 - n_1) - \alpha n_1 n_2$$
$$\frac{dn_2}{d\tau} = \gamma n_2(1 - n_2) - \beta n_1 n_2.$$