MATH 420 Homework #5 SOLUTIONS

1. Suppose $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ is a vector in $\mathbb{R}^2$ and $\dot{\mathbf{x}} = \mathbf{J}\,\mathbf{x}$, where $\mathbf{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and elements a, b, c, $d \in \mathbb{R}$.

   (a) Prove that the general equations for the eigenvalues of a 2x2 matrix are

$$\lambda_i = \frac{1}{2}\left(Tr(J) \pm \sqrt{Tr(J)^2 - 4\,Det(J)}\right)$$

   using the definition of λ_i as solutions to $\det(\mathbf{J} - \lambda \mathbf{I}) = 0$. This can be done by hand or using a computer algebra system (e.g., Maxima; but don't just use `eigenvalues()`).

   Answer: Solving for the roots (λ_i) of

$$\det(\mathbf{J} - \lambda\,\mathbf{x}) = \lambda^2 - (a+d)\lambda + a\,d - b\,c = \lambda^2 - Tr(\mathbf{J})\lambda + Det(\mathbf{J}),$$

   the quadratic equation gives that λ_1 and λ_2 are

$$\lambda_1 = \frac{1}{2}\left(Tr(J) - \sqrt{Tr(J)^2 - 4\,Det(J)}\right), \qquad \lambda_2 = \frac{1}{2}\left(Tr(J) + \sqrt{Tr(J)^2 - 4\,Det(J)}\right).$$

   (b) The associated eigenvectors satisfy $\mathbf{J}\,\mathbf{v}_i = \lambda_i\,\mathbf{v}_i$, and can be computed as follows:

$$\begin{array}{lll} \text{If } c \neq 0: & \mathbf{v}_1 = \begin{bmatrix} \lambda_1 - d \\ c \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \lambda_2 - d \\ c \end{bmatrix} \\ \text{If } b \neq 0: & \mathbf{v}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} b \\ \lambda_2 - a \end{bmatrix} \\ \text{If } c = b = 0: & \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}$$

   Compute eigenvalues & eigenvectors for the following cases using the formulas above.

   (i) $a = 1/10$, $b = -1$, $c = 1/2$, $d = -4$.
   Answer:

$$(\lambda_1, \mathbf{v}_1) = \left(-\frac{39 + \sqrt{1481}}{20},\ \begin{bmatrix} -\frac{\sqrt{1481}-41}{20} \\ 1/2 \end{bmatrix}\right) \quad (\lambda_2, \mathbf{v}_2) = \left(-\frac{39 - \sqrt{1481}}{20},\ \begin{bmatrix} \frac{\sqrt{1481}+41}{20} \\ 1/2 \end{bmatrix}\right)$$

   (ii) $a = 1$, $b = 1$, $c = -1$, $d = -2$.
   Answer: Let φ be the golden ratio (i.e., $\varphi = (1 + \sqrt{5})/2 \approx 1.6180$), then

$$(\lambda_1, \mathbf{v}_1) = \left(-\varphi,\ \begin{bmatrix} 2 - \varphi \\ -1 \end{bmatrix}\right) \quad (\lambda_2, \mathbf{v}_2) = \left(\varphi - 1,\ \begin{bmatrix} 1 + \varphi \\ -1 \end{bmatrix}\right)$$

(iii) $a = -1$, $b = 1$, $= -1$, $d = -1$.
Answer: Using the notation $\mathbf{i} = \sqrt{-1}$, then

$$(\lambda_1, \mathbf{v}_1) = \left(-1 - \mathbf{i},\ \begin{bmatrix} -\mathbf{i} \\ -1 \end{bmatrix}\right) \qquad (\lambda_2, \mathbf{v}_2) = \left(-1 + \mathbf{i},\ \begin{bmatrix} \mathbf{i} \\ -1 \end{bmatrix}\right).$$

(c) Use those results to determine whether or not the origin $(0, 0)$ is a *stable* or *unstable equilibrium* of the system $\dot{\mathbf{x}} = \mathbf{J}\,\mathbf{x}$. If the origin is a *saddle* (i.e., it's eigenvectors have positive and negative real parts) state which are the stable and unstable vectors (aka the start of the stable and unstable *manifolds* of the saddle).

Answer: (i) Stable, since $\text{Re}(\lambda_i) < 0$, for $i = 1, 2$.
(ii) Unstable, since $\lambda_2 > 0$. (An *unstable saddle* since $\lambda_1 < 0$)
(iii) Stable, since $\text{Re}(\lambda_i) = -1$, for $i = 1, 2$.

(d) For each case in part (b), find the eigenvalues and eigenvectors of $\mathbf{J}$ using the `matrix()` and `eigen()` functions in R.

```r
J = matrix(c(a = 1/10, b = -1, c = 1/2, d = -4), 2, 2, byrow = TRUE)
eigen(J)

## $values
## [1] -3.87418814 -0.02581186
##
## $vectors
##            [,1]      [,2]
## [1,] 0.2440174 0.9921784
## [2,] 0.9697709 0.1248278

J = matrix(c(a = 1, b = 1, c = -1, d = -2), 2, 2, byrow = TRUE)
eigen(J)

## $values
## [1] -1.618034  0.618034
##
## $vectors
##             [,1]       [,2]
## [1,] -0.3568221  0.9341724
## [2,]  0.9341724 -0.3568221

J = matrix(c(a = -1, b = 1, c = -1, d = -1), 2, 2, byrow = TRUE)
eigen(J)

## $values
## [1] -1+1i -1-1i
##
## $vectors
##                         [,1]                    [,2]
## [1,] 0.7071068+0.0000000i 0.7071068+0.0000000i
## [2,] 0.0000000+0.7071068i 0.0000000-0.7071068i
```

BONUS: Use Maxima or some other symbolic software to calculate a general formula for the eigenvalues of a 3x3 matrix (4x4?), and **submit the results electronically**.

Answer: Similar to the 2x2 case, this question is essentially asking us to find a formula for the roots of a cubic polynomial whose coefficients are functions of the entries of a 3x3 matrix. This general formula is not a simple formula! See the provided wxMaxima script for details.

2. In practice, we don't always need to find eigenvalues to determine equilibrium stability. The Routh-Hurwitz Criteria are a set of necessary and sufficient conditions for whether or not the roots of a polynomial have negative real part. You may recall that the **characteristic equation** of an $n \times n$ matrix $\mathbf{A}$ is the n^{th}-order polynomial $p(x) = \det(A - x\,I)$, and its roots are by definition the eigenvalues of $\mathbf{A}$.

Routh-Hurwitz Criteria

All roots of the polynomial (with real coefficients c_i)

$$p(x) = c_n + c_{n-1}\,x + \cdots + c_1\,x^{n-1} + x^n$$

have negative real parts if and only if the determinants of all the corresponding Hurwitz matrices are positive. This result provides an algorithm for computing stability criteria, which gives these equivalent conditions for small values of n:

$$\begin{aligned} n &= 2 \qquad c_i > 0 \\ n &= 3 \qquad c_i > 0,\ c_1\,c_2 > c_3 \\ n &= 4 \qquad c_i > 0,\ c_1\,c_2\,c_3 > c_3^2 + c_1^2\,c_4 \end{aligned}$$

Further reading: See Meinsma (1995), Gantmacher (1989), or Ch. 4 of *Introduction to Mathematical Biology* by Allen (2007).

The Lorenz equations are a simplification of a fluid dynamics model, that were derived to illustrate chaotic dynamics by Ed Lorenz in 1963.

$$\dot{x} = \sigma\,(y - x) \tag{1a}$$
$$\dot{y} = r\,x - y - x\,z \tag{1b}$$
$$\dot{z} = x\,y - b\,z \tag{1c}$$

Use the Routh-Hurwitz criteria to find stability conditions (i.e., conditions on the values of parameters $\sigma > 0$, $b > 0$, $r > 1$) for the equilibrium point

$$(x_*,\ y_*,\ z_*) = \left(\sqrt{b\,r - b},\ \sqrt{b\,r - b},\ r - 1\right)$$

by completing the following steps:

(a) Find the Jacobian for a general point (x, y, z) for equations (1a-1c). This can be done by hand, or using a computer algebra system (e.g. Maxima).

Answer:

$$\mathbf{J} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

(b) Evaluate the Jacobian at the given equilibrium point (x_*, y_*, z_*), then find its characteristic equation and the coefficients c_1, c_2, and c_3.

Answer:

$$\mathbf{J}_* = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b \cdot r - b} \\ \sqrt{b \cdot r - b} & \sqrt{b \cdot r - b} & -b \end{bmatrix}$$

(c) Use the Routh-Hurwitz criteria to find parameter conditions for the stability of the given equilibrium point.

Answer: Finding the characteristic polynomial and dividing by the leading coefficient (so that the x^3 term has coefficient 1) yields

$$x^3 + (\sigma + b + 1) \cdot x^2 + (b \cdot \sigma + b \cdot r) \cdot x + 2 \cdot b \cdot r \cdot \sigma - 2 \cdot b \cdot \sigma = 0$$

Therefore, $c_3 = 2b\, r\, \sigma - 2b\, \sigma$, $c_2 = b\, \sigma + b\, r$ and $c_1 = \sigma + b + 1$ applying the criteria above, the Routh-Hurwitz criteria are that each coefficient is positive (which is always true for the given constraints $r > 1$, $b, \sigma > 0$) and that

$$(\sigma + b + 1)(\sigma + r)b > 2b\, \sigma\, (r - 1)$$

which reduces to

$$(\sigma + b + 1)(\sigma + r) > 2\sigma\, (r - 1).$$

Hence, the given equilibrium point is stable whenever (1) it exists in the positive real orthant (i.e., x_*, y_*, and z_* are all positive real numbers) and (2) whenever

$$(\sigma + b + 1)(\sigma + r) > 2\sigma\, (r - 1).$$