MATH 420 Homework #4 — Solutions

1. In last week's homework, it was given that the 1-dimensional linear ODE

$$\frac{dN}{dt} = \lambda\, N$$

has the solution

$$N(t) = N_0 \exp(\lambda\, t).$$

Assuming $N \neq 0$, this can be shown by the separation of variables method from calculus. Dividing both sides by N and integrating over time interval $[0, t]$ yields

$$\begin{aligned}
\int_0^t \frac{dN/dt}{N} dt &= \int_0^t \lambda\, dt \\
\log(N(t)) - \log(N_0) &= \int_0^t \lambda\, dt, \qquad \text{where } N_0 \equiv N(0) \\
\log(N(t)) - \log(N_0) &= \lambda\, t
\end{aligned}$$

and by adding $\log(N_0)$ to each side and exponentiating, we get

$$N(t) = N_0 \, \exp(\lambda\, t)$$

It is rare, in practice, that we can explicitly solve ODE models and find analytical solutions. However, it is a possibility that should not be overlooked! For example, some (e.g., see http://press.princeton.edu/chapters/s03_8709.pdf, pg 99, eq. 3.4.5) have claimed that the theta-logistic model (below) has no closed-form solution for $N(t)$.

$$\frac{dN}{dt} = r\, N \left(1 - \left(\frac{N}{K}\right)^{\theta}\right) \tag{1}$$

Prove this claim false by using separation of variables, as above, to find an expression for $N(t)$ that is a solution to the theta-logistic equation (eq. (1)).

(**Hint:** Look up the steps to finding a solution to the logistic equation, and mimic those. After moving all of the N terms to one side of the equality, do a partial fractions decomposition then integrate. Recall what the derivative $\frac{d}{dx}\log(f(x))$ looks like, and use Maxima as needed.)

Answer: Separating variables gives

$$\frac{dN}{N\left(1 - \left(\frac{N}{K}\right)^{\theta}\right)} = r\, dt.$$

Partial fraction decomposition gives

$$\frac{dN}{N} + \frac{N^{\theta-1}/K^{\theta}\, dN}{1 - \left(\frac{N}{K}\right)^{\theta}} = r\, dt$$

Recalling that the derivative of $\log(U)$ is dU/U, and recalling the antiderivative of U^{n-1} is U^n/n, integrating both sides of the above equation over $[0,t]$ yields

$$\log(N)\Big|_{N_0}^{N(t)} - \frac{1}{\theta}\log\left(1-\left(\frac{N}{K}\right)^{\theta}\right)\Big|_{N_0}^{N(t)} = r\,t$$

Exponentiating both sides, and using properties of log() gives

$$\frac{N(t)}{N_0}\frac{\left(1-\left(\frac{N_0}{K}\right)^{\theta}\right)^{1/\theta}}{\left(1-\left(\frac{N(t)}{K}\right)^{\theta}\right)^{1/\theta}} = \exp(r\,t)$$

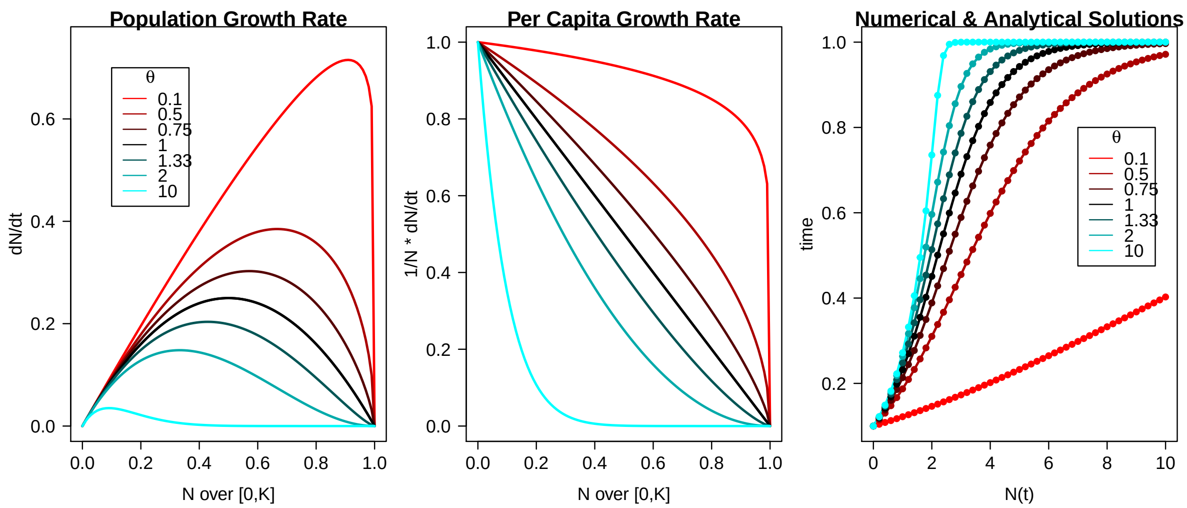
Raising both sides to power θ brings all $N(t)$ terms into the form $N(t)^{\theta}$

$$\frac{N(t)^{\theta}}{N_0^{\theta}}\frac{\left(1-\left(\frac{N_0^{\theta}}{K^{\theta}}\right)\right)}{\left(1-\left(\frac{N(t)^{\theta}}{K^{\theta}}\right)\right)} = \exp(\theta\,r\,t)$$

and so solving for $N(t)^{\theta}$ and taking that solution to the $1/\theta$ power yields

$$\begin{aligned} N(t) =& \frac{K\,N_0}{\left(N_0^{\theta} - \exp(-\theta\,r\,t)(N_0^{\theta}-K^{\theta})\right)^{1/\theta}} \\ =& \frac{K}{(1+\exp(-\theta\,r\,t)((K/N_0)^{\theta}-1))^{1/\theta}} \end{aligned}$$

Take a look at that the second term in the denominator. Note that if we start at the equilibrium value K (i.e., if $N_0 = K$) then $N(t) = K$ (as it should!). If not, then that second term goes to zero as $t \to \infty$ according to exponential decay, but with a rate that is either faster than ($\theta > 1$) or slower than ($\theta < 1$) a standard logistic equation ($\theta = 1$).



2. In class (see website for slides) we discussed the concepts of **state space** and **parameter space** for dynamical systems models. We can also categorize differential equations models by whether they are **linear** or **non-linear**, and by whether they are **autonomous** or **non-autonomous**. Look up the definitions of these terms, and for each model below, give (i) the dimension of their state space, (ii) the dimension of their parameter space, (iii) whether they are linear or non-linear, and (iv) whether they are autonomous or non-autonomous.

(a) $N(t) \in \mathbb{R}_0^+$ for all t, $r \in \mathbb{R}$.

$$\frac{dN(t)}{dt} = r\,N(t)$$

State Space: One dimensional ($N(t)$); $\mathbb{R}_0^+$.
Parameter Space: One dimensional (r); $\mathbb{R}$.
$f(N)$ is **linear**. The model is **autonomous**.

(b) $u(\tau) \in \mathbb{R}_0^+$ for all τ.

$$\frac{du}{d\tau} = (1-u)\,u$$

State Space: One dimensional (u); $\mathbb{R}_0^+$. **Parameter Space:** no parameters!
$f(u)$ is **non-linear**. The model is **autonomous**.

(c) $N(t) \in \mathbb{R}_0^+$ for all t; $r \in \mathbb{R}^+$, $K(t)$ a positive-valued differentiable function.

$$\frac{dN}{dt} = r\,x\,(1 - N/K(t))$$

State Space: One dimensional ($N(t)$); $\mathbb{R}_0^+$.
Parameter Space: 1-D (r); $\mathbb{R}^+$ (unless parameters of $K(t)$ augment that space).
$f(N)$ is **non-linear**. The model is **non-autonomous**.

(d) As above, with parameters K_∞, $\theta \in \mathbb{R}^+$.

$$\begin{aligned}\frac{dN}{dt} &= r\,N\left(1 - (N/K)^\theta\right)\\ \frac{dK}{dt} &= K_\infty - K\end{aligned}$$

State Space: 2 dimensional (N, K); $\mathbb{R}_0^+ \text{x}\, \mathbb{R}_0^+$. **Parameter Space:** 3-D (r, θ, K_∞)
$f(N, K)$ is **non-linear**. The model is **autonomous**[*] [*Corrected on 10/7/2015].

(e) $x(t) \in \mathbb{R}_0^+$ for all t; m constant.

$$\frac{dx}{dt} = m^3\,x$$

State Space: $\mathbb{R}_0^+$. **Parameter Space:** 1D. The model is **linear** and **autonomous**.