1. Use the R code provided in class (on the website) to find the input values that maximize the following functions. Do this using two methods: First, find the values that maximize f *analytically* (i.e., using calculus) then do it *computationally* (e.g., using `optimize` or `optimx` in R). How well the computationally and analytically derived values agree?

a. Let $a = 1$, $b = 2$, and $c = 0.5$. For $x > 0$, maximize

$$f(x) = \frac{x}{a + b\,x + c\,x^2}$$

Answer: Solving

$$f'(x) = \frac{(a - cx^2)}{(c^2x^4 + 2bcx^3 + (2ac + b^2)x^2 + 2abx + a^2)} = 0$$

positive a, b, c and x implies the unique positive solution is given by $a - cx^2 = 0$, thus

$$x = \sqrt{\frac{a}{c}} = \sqrt{2}.$$

Using R,

```
> f.1a = function(x) { x/(1+2*x+0.5*x^2) }
> optimize(f.1a,c(0,50),maximum=TRUE)
$maximum
[1] 1.414212

$objective
[1] 0.2928932
```

b. Maximize f over all x, $y \in \mathbb{R}$, where

$$f(x, y) = e^{-x^2-4y^2+4x-4}$$

Answer: Solving $\nabla f(x, y) = 0$ gives

$$(4 - 2x)\,f(x, y) = 0$$

$$-8y\,f(x, y) = 0.$$

Therefore, since $f(x, y) > 0$ for all x and y, the only solution is

$$x = 2, \; y = 0.$$

Using R, we get approximately the same answer after picking a better objective function:

```
> # Since the max of exp(f(x)) occurrs at the max of f(x),
> # let us find the min of -f(x)
> f.1b = function(z) {
+     x=z[1]
+     y=z[2]
+     return(-(-x^2-4*y^2+4*x-4))
+ }
> optim(c(0,0),f.1b)
$par
[1] 2.000168e+00 5.874413e-08

$value
[1] 2.808492e-08

$counts
function gradient
      65       NA

$convergence
[1] 0
```

c. Minimize f over all w, x, y, $z \in \mathbb{R}$, where

$$f(w, x, y, z) = \sqrt{17\, z^2 + \pi\, y^2 + \sqrt{2}\, x^2 + e\, w^2 + 1}$$

Answer: As above, solving $\nabla f(x, y) = 0$ gives

$$[w, x, y, z] = [0, 0, 0, 0]$$

Using R, we again can obtain a nearly equal answer:

```
> f = function(v) {
+     w=v[1];  x=v[2];  y=v[3];  z=v[4];
+     return(sqrt(17*z^2+pi*y^2+sqrt(2)*x^2+exp(1)*w^2+1))
+ }
> optim(c(2,2,2,2),f)
$par
[1]  2.197440e-04  3.604763e-04  9.642455e-05 -1.483529e-05

$value
[1] 1

$counts
function gradient
     379       NA

$convergence
[1] 0
```

2. Use LaGrange Multipliers (see section 2.2) to maximize $x+y$ when x and y are constrained to the circle $x^2+y^2=4$.

   Answer: Solving $\nabla f(x,y) = \lambda g(x,y)$ gives that

   $$x = \frac{1}{(2\lambda)}, y = \frac{1}{(2\lambda)}$$

   Under the given constraint, this implies $\lambda = \pm 1/2^{3/2}$ which, when substituted into the equation above yields the two optima (the latter being the maximum)

   $$x = -\sqrt{2},\ y = -\sqrt{2}$$

   $$x = \sqrt{2},\ y = \sqrt{2}$$

   Bonus: Use polar coordinates to solve #2 as an unconstrained optimization problem.

   Setting

   $$x = r\ \cos(\theta), \quad y = r\ \sin(\theta)$$

   the given constraint reduces to $r^2 = 4$, i.e., $r = 2$. Therefore, the optimization problem is reduced to a single dimension and it only remains to maximize

   $$x + y = 2(cos(\theta) + \sin(\theta)).$$

   Analytically, we can show this occurs where $\cos(\theta) = \sin(\theta)$ and both are positive, i.e., at $(\sqrt{2}, \sqrt{2})$. Computationally, we find

```
> obj=function(z){ 2*sin(z)+2*cos(z)}
> output=optimize(obj,c(0,2*pi),maximum=TRUE)
> theta=output$maximum
> x=2*cos(theta)
> y=2*sin(theta)
> c(x,y)
[1] 1.414215 1.414213
```