Dynamical Systems
Week 6 – Monday

Mathematical Modeling (Math 420/620)

Paul J. Hurtado

28 Sept, 2015

EECB Colloquium

High temperatures and the natural history of an impending extinction

Craig Benkman, University of Wyoming

2:30-3:30pm, **Thursday** (10/1) in **DMSC 103**

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Dynamic Model (ODE) Basics

Suppose $\mathbf{x} \in \mathbb{R}^n$, functions $f = [f_1, f_2, \ldots, f_n]$ are *smooth*[1], and

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x_0}.$$

[1]Continuous partial derivatives near $\mathbf{x_0}$ guarantee existence, uniqueness of solutions.

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State Variables:	$\mathbf{x} = [x_1, x_2, \ldots, x_n]$
Initial Conditions:	$\mathbf{x_0}$
State Space:	$S \subseteq \mathbb{R}^n$
Vector Field:	f
Parameter Space:	(Ex) $\mathbb{R}^{n^2}$ if f is a full linear system.
Trajectory/Orbit:	Solutions $\mathbf{x}(t)$ to the above IVP.

See also: order, (non)autonomous, (non)homogenious

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Equilibria

Trajectories are often categorized by **qualitative properties** (e.g. steady-state vs. cycling vs. chaos) of their **asymptotic behavior** (i.e., what do solutions look like as $t \to \infty$?).

Equilibrium solutions are the natural place to begin studying those asymptotic properties.

Definition

An **equilibrium** of

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

is any *constant* solution $\mathbf{x}(t) = \mathbf{x}_*$ which therefore satisfies

$$f(\mathbf{x}_*) = 0.$$

Equilibria

Find all equilibrium solutions to each of the following ODEs:

$$
\begin{aligned}
1.\quad \frac{dN}{dt} =& r\,N \\
2.\quad \frac{dx}{dt} =& K - x \\
3.\quad \frac{dx}{dt} =& x\,(K - x) \\
4.\quad \frac{dx}{dt} =& r\,x\,(1 - \frac{x}{K}) \\
5.\quad \frac{dx}{dt} =& x\,(1 - x)(a - x) \\
6.\quad \frac{dx}{dt} =& \sin(x)
\end{aligned}
$$

Example 4.1 – Two-species Competition

Goal: When can the two tree species coexist?

$$\dot{H} = r_H\, H - a_H\, H^2 - b_H\, S\, H$$

$$\dot{S} = r_S\, S - a_S\, S^2 - b_S\, H\, S$$

State Variables: (State Space is non-negative orthant in $\mathbb{R}^2$)

$H(t),\ S(t)$ - Hardwood & Softwood population size (tons/acre)

Rates: (Units are tons/acre/year)

$g_H(t) = r_H\, H - a_H\, H^2$ Hardwood growth rate

$g_S(t) = r_S\, S - a_S\, S^2$ Softwood growth rate

$c_H(t) = b_H\, S\, H$ - Hardwood loss rate

$c_S(t) = b_S\, S\, H$ - Softwood loss rate

Parameters: intrinsic growth rate r_i, *intraspecific* competition coefficients a_i, and *interspecific* competition coefficient b_i.

Question: Suppose $\mathbf{x}_*$ is an equilibrium solution to

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

and assume we perturb our initial condition to be ϵ-close to that value (i.e., let $\mathbf{x_0} \approx \mathbf{x}_*$).

Then does that trajectory converge to (or diverge from, or stay near) the equilibrium value $\mathbf{x}_*$?

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Answer: Conduct a stability analysis of the equilibrium $\mathbf{x}_*$.

Stability Concepts

1. We say $\mathbf{x}_*$ is **locally asymptotically stable (LAS)** (or sometimes just *locally stable* or *attracting*) if all nearby trajectories converge to $\mathbf{x}_*$ (i.e., $\mathbf{x}(t) \to \mathbf{x}_*$ as $t \to \infty$).

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2. We call $\mathbf{x}_*$ **globally asymptotically stable (GAS)** (or ...) if all trajectories converge to $\mathbf{x}_*$.
3. We say $\mathbf{x}_*$ is **Lyapunov Stable** if trajectories that start near it stay near $\mathbf{x}_*$.
4. We call $\mathbf{x}_*$ **neutrally stable** if it is Lyapunov Stable but not attracting.

Phase Space & 1-D Vector Fields

Phase Space: Horizontal axis x, vertical axis $\frac{dx}{dt}$.

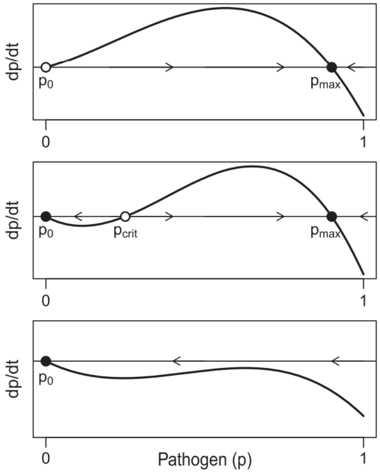
Bacterial growth model from Hurtado, 2012.

$$\frac{dp}{dt} = r\,p\,(1-p) - \frac{k\,p}{\mu + p}$$

Where

$$p_{crit} = \frac{\left((1-\mu) + \sqrt{(1+\mu)^2 - \frac{4}{r}k}\right)}{2}$$

$$p_{max} = \frac{\left((1-\mu) + \sqrt{(1+\mu)^2 - \frac{4}{r}k}\right)}{2}$$



Phase Space & 1-D Vector Fields

Sketch the *phase portrait* for each of the following, and use it to determine the stability of each equilibrium point:

$$
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